Linear Programming Primer

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A problem of the form: choose $x_1, x_2, \dots, x_n$

to minimize $c_1 x_1 + \dots + c_n x_n$

subject to: $a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \geq b_1$

$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \geq b_2$

$\vdots$

$a_{m1} x_1 + \dots + a_{mn} x_n \geq b_m$

where c_i's, a_{ij}'s, & b_j's are given constants

(i.e., minimize a linear function of the variables $x_1, \dots, x_n$ subject to m linear constraints)

is called a linear program.

Note that 'min' can be replaced by 'max',

$\geq$ by $\leq$

$=$ by $\leq, \geq$

Eg.:

$$\begin{array}{rcl} \max & -5x_1 + 2x_2 & \\ x_1 + x_2 & \geq & 4 \\ 2x_1 - 5x_2 & \geq & 0 \\ -x_1 - x_2 & \geq & -7 \end{array} \quad \equiv \quad \begin{array}{rcl} \min & 5x_1 - 2x_2 & \\ -x_1 - x_2 & \leq & -4 \\ 2x_1 - 5x_2 & \geq & 0 \\ x_1 + x_2 & \leq & 7 \end{array}$$

A feasible solution to a linear program is an assignment of values to the variables that satisfies all the constraints

An optimal solution is feasible, & minimizes the objective among all feasible solutions.

For the LP above, $(4,0)$ is a feasible solution, w/ objective value -20.

The optimal value is $\left(\frac{20}{7}, \frac{8}{7}\right)$, w/ objective value -12.

An LP can be written in matrix form:

(P) $\min \; c^T x$ where $c, x \in \mathbb{R}^n$

s.t. $Ax \geq b$ $\quad A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

in the example above, $c = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 2 & -5 \\ -1 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$

Every LP has a dual linear program:

(D) $\max \; b^T y$

s.t. $A^T y = c$

$y \geq 0$

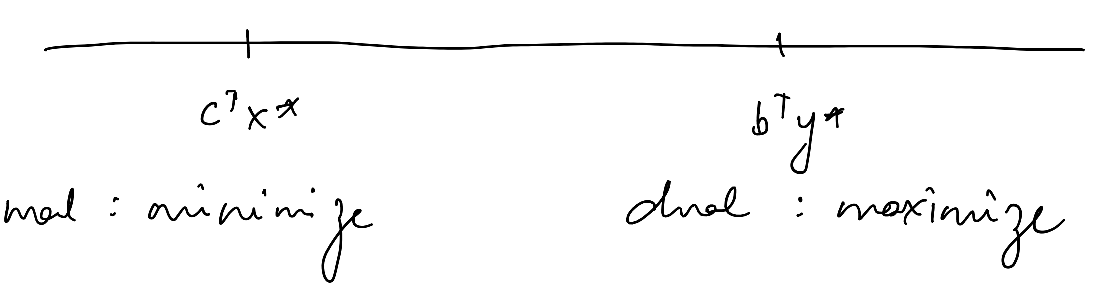
As can be expected, the dual of (D) is (P).

Strong Duality Theorem: If (P) & (D) are feasible, then the optimal primal value equals the optimal dual value. That is, if x^* & y^* are optimal values for the primal & dual respectively, then

$$c^T x^* = b^T y^*$$

Weak Duality Theorem: If x^* & y^* are feasible values for the primal & dual respectively, then

$$c^T x^* \geq b^T y^*$$



Finally, using interior-point methods, we can obtain an optimal solution to any LP in polynomial time.